

Financial Optimization Models in Data Networks*

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Abstract. In a competitive market investors in a data network need to give utmost considerations on profitability. They must have clear picture of the size, growth rate and demand for different services. However, the investors' budget may be limited, and therefore the speed at which the network is rolled out, must be carefully planned to ensure that they can meet profitability targets. We model first the roll out order as combinatorial optimization problems and then extend them as continuous optimization problems. We then implement these models in a practical problem. Numerical studies suggested that the optimization problems have multiple local minima. Therefore, a global optimization technique is used to obtain the global minimum for the continuous variable problem and a combinatorial optimization technique is used to solve the discrete variable problem. Optimal financial indicators are obtained to assess the commercial viability of the network. Finally, we demonstrate that the solution of these optimization problems can provide an investment policy to the investors in data networks.

Key words: data network, combinatorial optimization, global optimization, point of presence, rollout order, net present value, internal rate of return.

1. Introduction

When a data communication network is designed, three core issues must be considered. The first issue is the estimation of demand for various services on a geographical basis. Demand will depend on the concentration and the nature of the users. For instance, there will be more business users of telephone, fax and computers in urban areas than in semi-urban or rural areas. The second issue is the optimal placement of resources in the network. The network placement has to be done in an optimal way so as to minimize the cost of the network. Given a certain demand, the geographical distribution of the demand and the location of major landmarks (such as mountains or rivers), the resource planner must decide what capacity is required to meet the demand in the network and where to optimally place all the equipment in the network. For instance, should it be required to

*This network is a combined telephone and data network such as VIP (Voice over Internet Protocol).

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install a fibre optic backbone, where should the fibre be routed and where should the equipment be placed? The third issue is the profitability of the network. Once the demand for services is known, it is possible to determine the revenue that can be generated from these services. Combining this with the cost of the roll-out, the profitability of the network can be determined. We have addressed the first and the second issues in references (Ali et al., 2003; Ali, 2006). The third issue is considered in this paper. The purpose of this research, therefore, is to optimize the rate and order at which a network is rolled out in order to maximize the profit.

In Section 2, optimization models involving the rollout order are derived. Section 3 presents an example of a data network. Section 4 presents the algorithms used in the optimization of a data network problem. Section 5 presents the results of the optimization and demonstrates the usefulness of the models. Conclusions are given in section 6.

2. Financial Optimization

In this section, we show how the rollout order of a data network can be modeled as optimization problems. We model first the rollout order as combinatorial optimization problems which are then extended to continuous optimization problems. Formulations of the optimization problems are based on costs and revenues, and the internal rate of return.

The investors in a data network want to make their investment decisions based on what their investment policy is worth to them now. Therefore, before the implementation of the network, the investors decide on a specific time frame in which they would like to make a profit. We refer to this time frame as the ‘payback period’. We derive the profitability of the network within the payback period, although there is no reason why the investors cannot make profits beyond this period. We use the net present value (npv) and the internal rate of return (*irr*) (Sydsaeter and Hammond, 1997; Weston, 2001) as measures of profitability, as they are complementary measures of discounted cash flow (Weston, 2001). The npv and *irr* together give better analysis than either one of them. The interest rate (z_1) and the inflation rate (z_2) are taken to be constant over the payback period. In a stable economy these rates do not vary drastically over a short payback period. Profits from a network depend on the size of the network, the number of users and their growth rate (z_3) (growth rate of demand for services). We consider the network up to the level of the point of presence (PoP). Thus all financial calculations are carried out at the PoP level. We assume that there are n number of PoPs and PoP i ($i = 1, 2, \dots, n$) has a bandwidth capacity of b_i megabits per second (mbps), both ways. We refer to a connection between two PoPs as an edge. We assume that the initial assignment of bandwidth on the edges is sufficient to carry the traffic for the entire payback period.

There are revenues and costs (including the cable cost) associated with each PoP. The PoP i has an associated cable cost c^i . We refer to the reference (Ali, 2006) for more details on cable cost.

The cost is the currency value of any asset (including labour or knowledge) used in order to lay out and operate a network. The fibre optic technology that is installed at each PoP has a cost that can be divided into four components, namely (a) capital cost, (b) installation cost, (c) operating cost and (d) miscellaneous cost. We assume that the capital cost and the operating costs are dependent on the bandwidth capacity b_i of the i th PoP. The capital cost (K_i) includes the capital expenditure (offices and tents) and the cost of the instruments. The installation cost (I) is the cost for installation of the instruments at PoPs and the operating cost (O_i) is the cost for maintenance and operation, e.g. the human capital (salaries). The miscellaneous cost (M) is the incidental cost, e.g., the cost for a workshop and entertainments. We take into account the inflationary effect. Therefore, these costs are not fixed throughout the payback period.

We assume that there is an amount of traffic using the network per year, and that the tariffs are the main source of revenue for the investors. There are two types of tariffs: a fixed tariff, f_t , per year, per line (user) and a variable tariff, v_t , per megabytes. At each PoP i , we have a number of users n_i and usage u_i in gigabytes (gb), per year. Tariffs are not fixed throughout the payback period. The fixed tariff f_t and the variable tariff v_t increase at an annual rate of z_4 . The number of users n_i and usage u_i have a growth rate z_3 . Next, we derive the financial models before presenting the optimization problems.

2.1. FINANCIAL CALCULATIONS

At this point, we assume that a network will be installed within the payback period d and the installations are to be done at the beginning of each year. We also assume that the installation time is negligible, i.e. if the installation of a PoP starts at the beginning of a year then it will be completed within a short period of time so that the demand of the PoP will be satisfied within the same year. This implies that the revenues will be generated for d years from the PoPs installed at the beginning of the first year, and for one year from the PoPs installed at the beginning of the d -th year. Let us assume that the costs and revenues are calculated at equally spaced intervals. We will relax this assumption later in the paper.

We begin with the mathematical derivation of operating costs of PoPs. The operating cost is the annual running cost while the other costs are once-off. For instance, when a PoP is installed the money spent on capital, the installation and for miscellaneous reasons are spent once only. However, these costs vary with inflation. The operating cost also increases in direct proportion

to the inflation rate, since the cost of purchasing new materials or hiring people approximately follows the inflation rate. The operating cost can only be determined at the end of each year after a PoP has been installed. However, we assume that an estimate O_i of the operating cost per b_i megabits bandwidth of the i th PoP is known at the beginning of the first year. This assumption is based on the historical data.

Let x_i denote the installation year of the i th PoP, $x_i = 0, 1, 2, \dots, (d-1)$. The operating cost grows with the inflation rate, e.g., the cost increases to $O_i(1+z_2)^{x_i}$ at the x_i th year.

The total once-off cost is the addition of the capital cost, the installation cost and the cable cost, per PoP. If we install the i th PoP at the beginning of the first year of the payback period, the money spent is $K_i + I + M + c^i$. This cost at the beginning of the x_i th year will be

$$(K_i + I + M + c^i)(1+z_2)^{x_i}, \quad x_i \in \{0, 1, 2, \dots, (d-1)\}. \quad (1)$$

Therefore, if the i th PoP is installed at the beginning of x_i th year of the payback period, then the net present value, C_{x_i} , of the entire cost for this PoP is given by

$$C_{x_i} = (K_i + M + I + c^i)(1+z_2)^{x_i}(1+z_1)^{-x_i} + \sum_{j=(x_i+1)}^d O_i(1+z_2)^j(1+z_1)^{-j},$$

$$x_i \in \{0, 1, 2, \dots, (d-1)\}, \quad (2)$$

where the sum is the accumulated sum of the operating costs from x_i th to the d th year. We have taken the interest rate z_1 as the discount rate.

The revenue generated at the end of first year (year one) from the i th PoP installed at the beginning of the first year (i.e., year zero) is given by $n_i(1+z_3)^1 f_i(1+z_4)^1 + u_i(1+z_3)^1 v_i(1+z_4)^1$ (u_i converted to megabytes). The values for n_i , u_i , f_i and v_i are all known at the beginning of the first year and their incremental values are effective at the beginning of the second year. Therefore, the net present value of the revenues, R_{x_i} , generated at the i th PoP installed at the beginning of the x_i th year, is given by

$$R_{x_i} = \sum_{j=(x_i+1)}^d (n_i(1+z_3)^j f_i(1+z_4)^j + u_i(1+z_3)^j v_i(1+z_4)^j) (1+z_1)^{-j},$$

$$x_i \in \{0, 1, 2, \dots, (d-1)\}. \quad (3)$$

This can be rewritten as follows:

$$R_{x_i} = \sum_{j=(x_i+1)}^d (n_i f_i + u_i v_i)(1+z_4)^j(1+z_3)^j(1+z_1)^{-j},$$

$$x_i \in \{0, 1, 2, \dots, (d-1)\}. \quad (4)$$

We present the npv of the profit first at the PoP level and then at the network level. The npv of profit from the i th PoP installed at the beginning of the x_i th year is given by

$$P_{x_i}^v = R_{x_i} - C_{x_i}, \quad (5)$$

where $P_{x_i}^v$ is the npv of profit from the i th PoP ($i = 1, 2, \dots, n$). Finally, the npv of profit generated by all PoPs is given by

$$npv_{\text{pops}}^v = \sum_{i=1}^n R_{x_i} - C_{x_i}, \quad (6)$$

where npv_{pops}^v is the npv of the total profit from all PoPs.

2.1.1. Network Cost due to Subscription Fees

A network service provider has to pay an annual subscription fee to a telecommunication company for the use of its major infrastructure (telecommunication network). This annual fee X_0 , payable at the beginning of each year k ($k = 0, 1, \dots, (d-1)$), increases with inflation z_2 . However, the k th ($k = 1, 2, \dots, (d-1)$) year's subscription fee X_k is dependent on the usages U_k and U_{k-1} . The total usage U_j ($j = 0, 1, \dots, k$) is known at the beginning of the j th year. The usage U_j at the beginning of the j th year is the usage due to the PoPs installed at the beginning of j th year plus the usages resulting from the installation of PoPs in previous years. If $U_k \leq U_{k-1}$ then $X_k = X_0(1 + z_2)^k$ otherwise if $U_k > t_k \times U_{k-1}$ ($t_k > 1$) then $X_k = \lfloor t_k \rfloor \times X_0(1 + z_2)^k$, where $\lfloor t_k \rfloor$ is the largest integer not larger than t_k . This type of agreements are common between the telecommunication companies and the data network service providers. The npv of the subscription fees for a payback period d can be written as

$$S_f^v = \sum_{k=0}^{(d-1)} Y_k(1 + z_2)^k(1 + z_1)^{-k}, \quad (7)$$

where S_f^v is the npv of the subscription fees spread over the entire payback period, and $Y_k = \lfloor t_k \rfloor X_0$ for $k \geq 1$ and $Y_0 = X_0$.

2.1.2. The Internal Rate of Return

The internal rate of return (*irr*) is a discount rate at which the present value of a series of cashflow¹ (initial outlay and profits) is equal to zero.

¹If we denote k as the cashflow index for the entire network then $k = 0, 1, 2, \dots, d$. We consider the costs as the negative revenues and cashflows as the accumulated revenues at index k . Notice that the revenue from the network at $k = 0$ is always negative.

For a particular rollout order of the network there will be cashflows from various PoPs, each year. For example, there will be cashflows at years $x_i, x_i + 1, \dots, d$, due to the i th PoP installed at the x_i th year. If we denote our total cashflows at the beginning of k th year by $CF_k, k=0, 1, 2, \dots, d$, then the npv of the cashflows can be written as a function of irr as follows:

$$F(irr) = \sum_{k=0}^d CF_k (1 + irr)^{-k}. \quad (8)$$

The irr is the root of F or the solution of the equation $F(irr) = 0$. Therefore, irr is the discount rate for which the above npv is zero. The first cashflow is given by

$$CF_0 = -Y_0 - \sum_{i=1}^{n_0} (K_{0_i} + M + I + c^{0_i}), \quad (9)$$

where Y_0 is the subscription fee, n_0 is the number of PoPs installed in year zero (i.e., beginning of the first year), and c^{0_i} and K_{0_i} are cable and capital costs respectively for the 0_i th PoP installed in year zero. The k th cashflow is given by

$$\begin{aligned} CF_k = & -Y_k (1 + z_2)^k - \sum_{i=1}^{n_k} (K_{k_i} + M + I + c^{k_i}) (1 + z_2)^k \\ & + \sum_{j=0}^{(k-1)} \sum_{i=1}^{n_j} [(n_{j_i} f_i + u_{j_i} v_i) (1 + z_3)^k (1 + z_4)^k - O_{j_i} (1 + z_2)^k], \\ & k = 1, 2, \dots, (d-1), \end{aligned} \quad (10)$$

where the first term is the subscription fee and the second term is the once-off cost from all n_k PoPs installed at the beginning of the k th year. The first sum in the third term in (10) is the sum over preceding installation years and the second sum is the sum over the number of PoPs installed in each such years. The j_i is the index of the PoP installed in j th year. The final cashflow is given by

$$\begin{aligned} CF_d = & \sum_{j=0}^{(d-1)} \sum_{i=1}^{n_j} [(n_{j_i} f_i + u_{j_i} v_i) (1 + z_3)^d (1 + z_4)^d - O_{j_i} (1 + z_2)^d] \\ = & \sum_{i=1}^{n_0} [(n_{0_i} f_i + u_{0_i} v_i) (1 + z_3)^d (1 + z_4)^d - O_{0_i} (1 + z_2)^d] \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=1}^{n_1} [(n_{1_i} f_i + u_{1_i} v_i)(1 + z_3)^d (1 + z_4)^d - O_{1_i} (1 + z_2)^d] +, \dots, \\
& + \sum_{i=1}^{n_{(d-1)}} [(n_{(d-1)_i} f_i + u_{(d-1)_i} v_i)(1 + z_3)^d (1 + z_4)^d - O_{(d-1)_i} (1 + z_2)^d].
\end{aligned} \tag{11}$$

Notice that unlike (10), (11) does not have installation costs. We have used the Newton method (Nocedal and Wright, 1999) to obtain the root of F in (8).

2.2. COMBINATORIAL OPTIMIZATION PROBLEMS

We consider two optimization problems, namely A_P and B_P . The first problem, A_P , is to maximize the net present value of profits from the entire network. This problem is concerned with the maximization of the combined net present value of (6) and negative of (7). The variables of the optimization problem are the integer variables x_i taking values from the set $\{0, 1, 2, \dots, (d-1)\}$ for $i = 1, 2, \dots, n$. Let $x = (x_1, x_2, \dots, x_n)$ be a vector of discrete random variables, taking values from the set $\{0, 1, 2, \dots, (d-1)\}$. The objective function is given by

$$f_p(x_1, x_2, \dots, x_n) = -S_f^v + npv_{\text{pops}}^v, \tag{12}$$

where f_p is the net present value for the entire network. More precisely, the maximization problem can be stated as follows:

$$\begin{aligned}
\max_{x_i \in \{0, 1, 2, \dots, (d-1)\}} f_p(x_1, x_2, \dots, x_n) &= - \sum_{k=0}^{(d-1)} Y_k (1 + z_2)^k (1 + z_1)^{-k} \\
& + \sum_{i=1}^n \left(\sum_{j=(x_i+1)}^d [(n_i f_i + u_i v_i)(1 + z_3)^j (1 + z_4)^j (1 + z_1)^{-j} \right. \\
& \left. - O_i (1 + z_2)^j (1 + z_1)^{-j}] - (K_i + M + I + c^i)(1 + z_2)^{x_i} (1 + z_1)^{-x_i} \right), \\
& \text{subject to } g_{ij}(x_1, x_2, \dots, x_n) = x_{r_i} - x_{r_j} \leq 0 \\
& \text{for some } i, j \in \{1, 2, \dots, n\}, i \neq j.
\end{aligned} \tag{13}$$

The constraint g_{ij} in (13) implies that the installation of the r_i th PoP must precede that of the r_j th PoP, due to the link or connectivity between the PoPs. There may be several such constraints. S_f^v implicitly depends on x . One needs to calculate usages at U_k and U_{k-1} in order to calculate Y_k in (13).

The second problem, B_P , is the maximization of irr as a solution of $F(irr)=0$ in (8). The problem is to maximize the irr as a function of cashflows CF_k , $k=0, 1, \dots, d$, resulting from a rollout order. This problem can be defined as follows:

$$\begin{aligned} & \max \quad irr, \\ & \text{subject to } F(irr) = \sum_{k=0}^d CF_k(1+irr)^{-k} = 0 \\ & \text{and} \quad g_{ij}(x_1, x_2, \dots, x_n) = x_{r_i} - x_{r_j} \leq 0 \\ & \quad \text{for some } i, j \in \{1, 2, \dots, n\}, i \neq j, \end{aligned} \quad (14)$$

where CF_k are defined by (9)–(11). Clearly irr depends on the cashflows CF_k which in turn depend on the rollout order. Thus, both A_P and B_P are functions of the installation year. Which PoP will be installed at the beginning of which year (rollout order) will depend upon the quantity and the types of demands, costs and revenues they generate.

2.3. GENERALIZATIONS OF PROBLEMS A_P AND B_P

In the formulation of A_P and B_P , we assume that PoPs are installed at the beginning of each year of the payback period. We would like to relax this assumption for the purpose of generalizations. We assume that PoPs are installed at the beginning of any units of time. For example, installations can occur annually, semiannually, quarterly, monthly and so on. Let m be the frequency of installation per annum. Thus $m = 1, 2, 4, 12$ means PoPs are installed annually, semiannually, quarterly and monthly respectively. We further assume that the interest rate is discounted, and the inflation and the growth rates are compounded on the basis of frequency m of PoP installation. That is, if $m = 12$ is the frequency of installation per annum then the interest rate is discounted monthly. For example, if the interest rate is compounded m times per annum then the terminal value of a unit amount investment for d years will be

$$\left(1 + \frac{z_1}{m}\right)^{md}. \quad (15)$$

Under the above considerations, the combinatorial optimization problem A_P given by (13) has the following general form:

$$\begin{aligned} & \max_{x_i \in \{0, 1, 2, \dots, (md-1)\}} f_p(x_1, x_2, \dots, x_n) \\ & = -S_f^v + \sum_{i=1}^n \left(\sum_{j=(x_i+1)}^{md} \left[(n_i f_i + u_i v_i) \left(1 + \frac{z_3}{m}\right)^j \left(1 + \frac{z_4}{m}\right)^j \left(1 + \frac{z_1}{m}\right)^{-j} \right] \right) \end{aligned}$$

$$\begin{aligned}
& -O_i \left(1 + \frac{z_2}{m}\right)^j \left(1 + \frac{z_1}{m}\right)^{-j} \Big] - (K_i + M + I + c^i) \left(1 + \frac{z_2}{m}\right)^{x_i} \left(1 + \frac{z_1}{m}\right)^{-x_i} \Big) \\
& \text{subject to } g_{ij}(x_1, x_2, \dots, x_n) = x_{r_i} - x_{r_j} \leq 0 \\
& \text{for some } i, j \in \{1, 2, \dots, n\}, i \neq j.
\end{aligned} \tag{16}$$

Although the annual subscription fee is paid at the beginning of each year, it can be compounded m times per annum. Hence the problem A_P can be written as follows

$$\begin{aligned}
& \max_{x_i \in \{0, 1, 2, \dots, (md-1)\}} f_P(x_1, x_2, \dots, x_n) \\
& = - \sum_{k=0}^{(d-1)} Y_k \left(1 + \frac{z_2}{m}\right)^{mk} \left(1 + \frac{z_1}{m}\right)^{-mk} \\
& + \sum_{i=1}^n \left(\sum_{j=(x_i+1)}^{md} \left[(n_i f_t + u_i v_t) \left(1 + \frac{z_3}{m}\right)^j \left(1 + \frac{z_4}{m}\right)^j \left(1 + \frac{z_1}{m}\right)^{-j} \right. \right. \\
& \left. \left. - O_i \left(1 + \frac{z_2}{m}\right)^j \left(1 + \frac{z_1}{m}\right)^{-j} \right] \right. \\
& \left. - (K_i + M + I + c^i) \left(1 + \frac{z_2}{m}\right)^{x_i} \left(1 + \frac{z_1}{m}\right)^{-x_i} \right) \\
& \text{subject to } g_{ij}(x_1, x_2, \dots, x_n) = x_{r_i} - x_{r_j} \leq 0 \\
& \text{for some } i, j \in \{1, 2, \dots, n\}, i \neq j.
\end{aligned} \tag{17}$$

The optimization problem (13) is a special case of (17), these two problems are equal when $m=1$. In order to facilitate the understanding and to make the difference between the problems more explicit, we denote the problem in (17) by A_P^m and the problem in (13) by A_P^1 , as a special case when $m=1$.

A generalization, B_P^m , of (15) can similarly be defined by

$$\begin{aligned}
& \max \quad irr, \\
& \text{subject to } F(irr) = \sum_{k=0}^{md} CF_k \left(1 + \frac{irr}{m}\right)^{-k} = 0 \\
& \text{and } g_{ij} = (x_1, x_2, \dots, x_n) \leq 0 \quad \text{for some } i, j, i \neq j,
\end{aligned} \tag{18}$$

where

$$CF_0 = -Y_0 - \sum_{i=1}^{n_0} (K_{0_i} + M + I + c^{0_i}), \tag{19}$$

$$\begin{aligned}
CF_k = & -S_f - \sum_{i=1}^{n_k} (K_{k_i} + M + I + c^{k_i}) \left(1 + \frac{z_2}{m}\right)^k \\
& + \sum_{j=0}^{(k-1)} \sum_{i=1}^{n_j} \left[(n_{j_i} f_t + u_{j_i} v_t) \left(1 + \frac{z_3}{m}\right)^k \left(1 + \frac{z_4}{m}\right)^k - O_{j_i} \left(1 + \frac{z_2}{m}\right)^k \right], \\
& k = 1, 2, \dots, (md - 1),
\end{aligned} \tag{20}$$

where

$$S_f = \begin{cases} Y_{\lfloor \frac{k}{m} \rfloor} \left(1 + \frac{z_2}{m}\right)^k & \text{if } k \equiv 0 \pmod{m}, k = 1, \dots, (md - 1) \\ 0 & \text{otherwise.} \end{cases}$$

Finally,

$$CF_{md} = \sum_{j=0}^{(md-1)} \sum_{i=1}^{n_j} \left[(n_{j_i} f_t + u_{j_i} v_t) \left(1 + \frac{z_3}{m}\right)^{md} \left(1 + \frac{z_4}{m}\right)^{md} - O_{j_i} \left(1 + \frac{z_2}{m}\right)^{md} \right]. \tag{21}$$

The equations (10) and (11) are special cases of equations (20) and (21) respectively.

2.4. CONTINUOUS OPTIMIZATION PROBLEMS

We further relax the installation time of PoPs for the derivation of continuous optimization problems. In order to make the installation time x_i as continuous variables, we consider continuous compounding of all the rates (interest, inflation and growth rates). This means the frequency m tends to infinity. With continuous compounding, the terminal value in (15) is now $e^{z_1 d}$ where d can now be treated as real (Hull, 1995). The continuous version, A_p^∞ , resulted from taking limit $m \rightarrow \infty$ in (17), of A_p^m is now given by

$$\begin{aligned}
\max_{x_i \in [0, d]} f_p(x_1, x_2, \dots, x_n) = & - \sum_{k=0}^{(d-1)} Y_k \exp(k(z_2 - z_1)) \\
& + \sum_{i=1}^n \left[\sum_{x_i}^d (n_i f_t + u_i v_t) \exp(x_i(z_3 + z_4 - z_1)) \right. \\
& \times \left. 1 - \sum_{x_i}^d O_i \exp(x_i(z_2 - z_1)) \times 1 - (K_i + M + I + c^i) \exp(x_i(z_2 - z_1)) \right], \\
\text{subject to } & g_{ij}(x_1, x_2, \dots, x_n) = x_{r_i} - x_{r_j} \leq 0 \\
\text{for some } & i, j \in \{1, 2, \dots, n\}, i \neq j.
\end{aligned} \tag{22}$$

This can be written as

$$\begin{aligned}
\max_{x_i \in [0, d]} f_p(x_1, x_2, \dots, x_n) &= - \sum_{k=0}^{(d-1)} Y_k \exp(k(z_2 - z_1)) \\
&+ \sum_{i=1}^n \left[(n_i f_i + u_i v_i) \lim_{N \rightarrow \infty} \sum_{i=0}^N \left(\frac{d - x_i}{N} \right) \exp((z_3 + z_4 - z_1)) \right. \\
&\times \left(x_i + i \frac{d - x_i}{N} \right) - O_i \lim_{N \rightarrow \infty} \sum_{i=0}^N \left(\frac{d - x_i}{N} \right) \exp\left((z_2 - z_1) \left(x_i + i \frac{d - x_i}{N} \right) \right) \\
&\left. - (K_i + M + I + c^i) \exp(x_i(z_2 - z_1)) \right], \\
\text{subject to } g_{ij}(x_1, x_2, \dots, x_n) &= x_{r_i} - x_{r_j} \leq 0 \\
\text{for some } i, j \in \{1, 2, \dots, n\}, i \neq j. &
\end{aligned} \tag{23}$$

Taking limit $N \rightarrow \infty$ we get

$$\begin{aligned}
\max_{x_i \in [0, d]} f_p(x_1, x_2, \dots, x_n) &= - \sum_{k=0}^{(d-1)} Y_k \exp(k(z_2 - z_1)) \\
&+ \sum_{i=1}^n \left[(n_i f_i + u_i v_i) \int_{x_i}^d \exp((z_3 + z_4 - z_1)x) dx \right. \\
&\left. - O_i \int_{x_i}^d \exp((z_2 - z_1)x) dx - (K_i + M + I + c^i) \exp(x_i(z_2 - z_1)) \right], \\
\text{subject to } g_{ij}(x_1, x_2, \dots, x_n) &= x_{r_i} - x_{r_j} \leq 0 \\
\text{for some } i, j \in \{1, 2, \dots, n\}, i \neq j. &
\end{aligned} \tag{24}$$

Integrating out we get

$$\begin{aligned}
\max_{x_i \in [0, d]} f_p(x_1, x_2, \dots, x_n) &= - \sum_{k=0}^{(d-1)} Y_k \exp(k(z_2 - z_1)) \\
&+ \sum_{i=1}^n \left[\frac{(n_i f_i + u_i v_i)}{(z_3 + z_4 - z_1)} \left(\exp((z_3 + z_4 - z_1)d) - \exp((z_3 + z_4 - z_1)x_i) \right) \right. \\
&- \frac{O_i}{(z_2 - z_1)} \left(\exp((z_2 - z_1)d) - \exp((z_2 - z_1)x_i) \right) \\
&\left. - (K_i + M + I + c^i) \exp(x_i(z_2 - z_1)) \right], \\
\text{subject to } g_{ij}(x_1, x_2, \dots, x_n) &= x_{r_i} - x_{r_j} \leq 0 \\
\text{for some } i, j \in \{1, 2, \dots, n\}, i \neq j. &
\end{aligned} \tag{25}$$

The values Y_k in the first term in (25) are calculated from U_k and U_{k-1} by obtaining the nearest integer value to x_i for each PoP.

We now present the continuous version of the problem (18). The installation time for each PoP is now continuous. We consider each of these times as the time of cashflow, i.e. the cashflow index. There are $n + 1$ cashflow indexes due to n POPs. These include the endpoint of the payback period. Positions of n cashflow indexes change with the installation time. We consider the cashflows due to the subscription fee separately. There are therefore $(n + 1) + d$ cashflows in total. Using these cashflows we write the continuous version, B_P^∞ , of B_P^m by

$$\begin{aligned} & \max \quad irr, \\ & \text{subject to} \quad F(irr) = \sum_{i=1}^{(n+1)+d} CF_i \exp(-irrx_i) = 0 \\ & \text{and} \quad g_{ij}(x_1, \dots, x_n) = x_{r_i} - x_{r_j} \leq 0 \quad \text{for some } i, j, i \neq j, \end{aligned} \quad (26)$$

where x_{r_i} is the installation time of the r_i th PoP. Without loss of generality, we assume that

$$x_{r_1} < x_{r_2} < \dots < x_{r_n} < x_{r_{(n+1)}} = d.$$

The cashflow due to the PoP installed at the earliest time is given by

$$CF_1 = -(K_{r_1} + M + I + c^{r_1}) \exp(z_2 x_{r_1}). \quad (27)$$

Notice that the very first installation does not generate any revenues. The cashflow generated at the i th installation is given by

$$\begin{aligned} CF_i &= -(K_{r_i} + M + I + c^{r_i}) \exp(z_2 x_{r_i}) \\ &\quad - \sum_{j=1}^{i-1} \int_{x_{r_j}}^{x_{r_i}} [O_{r_j} \exp(z_2 x) - (n_{r_j} f_t + u_{r_j} v_t) \exp((z_3 + z_4)x)] dx, \\ &\quad i = 2, 3, \dots, n. \end{aligned} \quad (28)$$

The first term in (28) is the once-off cost due to the installation of the r_i th PoP and the second term is the accumulation of operating costs and revenues at x_{r_i} from all PoPs that are installed prior to the r_i th PoP. Integrating out we get

$$\begin{aligned} CF_i &= -(K_{r_i} + M + I + c^{r_i}) \exp(z_2 x_{r_i}) - \sum_{j=1}^{i-1} \left(\frac{O_{r_j}}{z_2} (\exp(z_2 x_{r_i}) - \exp(z_2 x_{r_j})) \right. \\ &\quad \left. + \frac{(n_{r_j} f_t + u_{r_j} v_t)}{(z_3 + z_4)} (\exp((z_3 + z_4)x_{r_j}) - \exp((z_3 + z_4)x_{r_i})) \right), \\ &\quad i = 2, 3, \dots, n. \end{aligned} \quad (29)$$

At the final cashflow index there is no installation and therefore the cashflow is given by

$$CF_{n+1} = - \sum_{j=1}^n \int_{x_{r_j}}^{x_{r_{(n+1)}}} [O_{r_j} e^{z_2 x} - (n_{r_j} f_t + u_{r_j} v_t) e^{(z_3 + z_4)x}]. \quad (30)$$

Integrating out we get

$$CF_{n+1} = - \sum_{j=1}^n \left[\frac{O_{r_j}}{z_2} (\exp(z_2 x_{r_{(n+1)}}) - \exp(z_2 x_{r_j})) + \frac{(n_{r_j} f_t + u_{r_j} v_t)}{(z_3 + z_4)} (\exp((z_3 + z_4)x_{r_j}) - \exp((z_3 + z_4)x_{r_{(n+1)}})) \right], \quad (31)$$

where $x_{(n+1)} = d$. The cashflows due to subscription fees are given by

$$CF_{(n+1)+1+j} = -Y_j \exp(z_2 j), \quad j = 0, 1, \dots, (d-1). \quad (32)$$

The cashflows in (32) are generated at the beginning of each year and they depend on the installation year of all PoPs. Both the problems A_P^∞ and B_P^∞ are n -dimensional where each continuous variable x_i is defined in $[0, d]$.

3. Application to a Data Network Problem

In this section, we consider a practical network problem to demonstrate the usefulness of the optimization models presented in the previous section. We show that the results of the optimization problems can assess the commercial viability and assist the investors in the network. The problem concerned here forms part of a network placement project involving a Southern African region of 500 kilometers (km) by 500 km, where an entire network has to be serviced (Alcatel, private communication). For more details of the network,² see the references (Ali et al., 2003; Ali, 2006). There are a total of 36 PoPs for the entire network. All PoPs have a bandwidth capacity of 100 megabits per second (mbps), both ways. All financial calculations are based on costs and revenues involved in the network. The South African currency, Rand (R), has been used in our calculation. We summarize various costs per PoP per year in Table 1. We take the cost per 100 km of cable as the unit cable cost. This unit cable cost is taken to be R16 000. We present the cable length for each PoP in Table 2, where the i th PoP is denoted by P_i . The i th PoP has an associated cable cost c^i which can be found using the data in Table 2. The entire region where the

²A description of the problem A_P^1 is also given in (Ali, 2006).

Table 1. Costs per 100 mbps bandwidth per PoP

Cost type	Amount in Rand (R)
Capital cost (K_i)	R1 200 000
Installation cost (I)	R350 000
Miscellaneous cost (M)	R50 000
Operating cost (O_i) per year	R2 250 000

Table 2. Cable length per PoP in kilometers

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
20	42.5	30	52.5	57.5	37.5	40	50	23
P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}
65	68	74.5	18	40	45	72.5	42.5	60
P_{19}	P_{20}	P_{21}	P_{22}	P_{23}	P_{24}	P_{25}	P_{26}	P_{27}
35	27.5	25	60	32.5	55	69	27.5	27
P_{28}	P_{29}	P_{30}	P_{31}	P_{32}	P_{33}	P_{34}	P_{35}	P_{36}
57	55	32	50	33	32.5	25	23.5	30

network has to be placed is divided into four different regions R_i , $i = 1, 2, 3$ and 4 (Ali et al., 2003; Ali, 2006). There are different numbers of PoPs that are to be placed in each region. At each PoP P_i , we have a number of users n_i and usage u_i in gigabytes (gb). We summarize the number of users and their yearly demand per PoP in Table 3. The last row in this table represents the total n_i and u_i per region. The amount of traffic using the network per year is (expected usage) is 36936 gigabytes. We use the fixed tariff $f_i = 30$ and the variable tariff $v_i = 3$ (in Rands).

Given the above demand, costs and revenues the problem is to obtain an optimal rollout order so as to maximize the profits over a five year payback period from the entire network. The six installation time constraints for the network problem are as follows (Alcatel, private communication).

$$\begin{aligned}
 x_{10} &\leq x_1, x_6, x_9, \\
 x_{12} &\leq x_{15}, \\
 x_{16} &\leq x_{18}, x_{19}.
 \end{aligned} \tag{33}$$

4. Global Optimization Algorithms

We used the simulated annealing (SA) algorithm (Aarts and Korst, 1989; Kirkpatrick, et al., 1983) to solve the discrete version of the optimization problem introduced in the previous section. For the continuous optimization problem, we used a modified version (MDE) (Kaelo and Ali, 2006) of

Table 3. Number of users and usage (gb) per PoP

R_1			R_2			R_3			R_4		
P_p	n_p	u_p	P_p	n_p	u_p	P_p	n_p	u_p	P_p	n_p	u_p
P_1	900	738.7200	P_{11}	810	1071.144	P_{16}	2400	3693.600	P_{25}	2100	2585.520
P_2	390	369.360	P_{12}	1860	4062.960	P_{17}	600	406.296	P_{26}	570	701.784
P_3	690	738.720	P_{13}	960	775.656	P_{18}	900	1108.080	P_{27}	630	775.656
P_4	600	1108.080	P_{14}	600	369.360	P_{19}	510	627.912	P_{28}	690	517.104
P_5	600	738.720	P_{15}	600	923.400	P_{20}	690	886.464	P_{29}	780	775.656
P_6	900	1108.080				P_{21}	570	701.784	P_{30}	810	960.336
P_7	300	554.040				P_{22}	630	775.656	P_{31}	840	849.528
P_8	450	590.976				P_{23}	420	517.104	P_{32}	600	664.848
P_9	810	886.464				P_{24}	720	849.528	P_{33}	840	627.912
P_{10}	2130	2954.880							P_{34}	450	590.976
									P_{35}	900	812.592
									P_{36}	750	517.104
Total	7770	9788.040		4830	7202.520		7440	9566.424		9960	10379.016

the differential evolution (DE) algorithm (Storn and Price, 1997). In this section we briefly present the idea of simulated annealing and differential evolution. More details on these methods can be found in the respective references.

4.1. SIMULATED ANNEALING

The SA algorithms are characterized by a point-wise search through the feasible region. Trial points are generated according to some distribution over the search region and a jump to a better point is always accepted. In order for the algorithm to be able to leave local minima, a jump to a worse point is also accepted but with a probability which decreases as the algorithm progresses. The technique for decreasing this probability, called the cooling schedule, is designed such that, using the theory of Markov chains, convergence to a global minimum is assured (Dekkers and Aarts, 1991).

In any implementation of simulated annealing, a cooling schedule must be specified. The initial temperature $\text{Temp}^{(0)}$ is usually large, so that most cost increasing trials are accepted and there is little chance of the algorithm zooming in on a local minimum. A scheme is then required for reducing the temperature. Finally a stopping criterion is required for the algorithm. The choice of a cooling schedule clearly has an important bearing on the performance of a SA algorithm. Here, we use the cooling schedule suggested by (Dekkers and Aarts, 1991).

The original simulated annealing algorithms are memoryless in the sense that no previous locations of the point are recorded (not even a possible visit at the global minimum). Therefore, in the SA process, it is possible

that during a particular stage, the procedure will visit the optimal solution, but due to an acceptance-rejection mechanism, it will move on from this solution and finish at a less than optimal solution. To overcome this difficulty, we add a memory f_p^* (*irr** for the irr problem) to the process (see the aspiration based simulated annealing (ASA) in (Ali and Storey, 1997)). If $\bar{x} = (x_1, x_2, \dots, x_n)$ is the current accepted point in the Markov chain then the next point \hat{x} of the inner loop (see the ASA algorithm for the definition of inner and outer loops) is generated within the search space with some distribution scheme. We accept \hat{x} if the following condition holds

$$\exp\left(\frac{f_p(\bar{x}) - f_p(\hat{x})}{\text{Temp}^{(k)}}\right) \geq \text{Ran}, \quad (34)$$

where Ran is a random number generated from the interval $[0, 1]$, $\text{Temp}^{(k)}$ is the temperature of the k th iteration of the outer loop. Therefore, at each stage corresponding to a fixed temperature, a Markov Chain is created for which some points will be accepted and others will be rejected. If a point generates a smaller function value than f_p^* (or *irr**) then the memory is updated. This is carried out at each step of the Markov chain. At the end, the last f_p^* is taken as the final solution to f_p . We now define the ASA algorithm with memory.

ALGORITHM 1: The ASA Algorithm

- Step 0. Initialization.** Calculate the initial temperature $\text{Temp}^{(0)}$ according to (Dekkers and Aarts, 1991), choose a random starting point $X^0 = \bar{x}$ and calculate $f_p(\bar{x})$. Initialize f_p^* with $f_p(\bar{x})$. Set $l = k = 0$.
- Step 1. Metropolis simulation in inner loop.** Generate the L points in the k th Markov chain according to the following rule:
- (a) Generate the new point $X^{l+1} = \hat{x}$.
- If

$$\exp\left(\frac{f_p(X^l) - f_p(X^{l+1})}{\text{Temp}^{(k)}}\right) \geq \text{Ran}$$

then accept the new point and update f_p^* , if necessary. Set $X^l = X^{l+1}$. Set $l = l + 1$ and if $l \leq L$ go to (a); otherwise go to Step 2.

- Step 2. Convergence.** If the stopping condition is satisfied then stop.
- Step 3. Outer loop.** Set $k = k + 1$ and $l = 0$. Reduce the temperature to $\text{Temp}^{(k+1)}$ and goto Step 1.

Point generation rule in SA: Central to the SA algorithm is its cooling schedule and we have used the cooling schedule suggested in (Dekkers and Aarts, 1991). Implementation of the cooling schedule requires setting some parameter values. We set these parameter values according to the

suggestions in (Dekkers and Aarts, 1991). At each temperature the ASA algorithm generates a Markov chain. The length of a Markov chain is the number of points generated in the Markov chain. A point here means a point, say \bar{x} in the solution space. Points in the Markov chain are generated both locally and globally. By locally we mean a new point X^{l+1} from the current accepted point X^l will be generated from some neighborhood of X^l . We refer to the reference (Lee et al., 2004) for the neighbourhood structure for the discrete variable \bar{x} . By globally we mean the components of X^{l+1} will be generated randomly within the search space. The method used for generating new points in the Markov chain involves the uniform distribution (75% globally) and a local technique (25% locally) as suggested in (Dekkers and Aarts, 1991). A random number $\text{Ran} \in (0, 1)$ is generated. If $\text{Ran} > 0.75$ then X^{l+1} is generated locally from the currently visited points; otherwise X^{l+1} is generated randomly (globally) for each Markov chain (Dekkers and Aarts, 1991).

The ASA algorithm is a memory based simulated annealing algorithm with a number of parameters of the cooling schedule. For the description of the parameters and their optimal values, see (Ali and Storey, 1997). Implementation of ASA to combinatorial optimization problems with parameter values can also be found in (Lee et al., 2004).

4.2. DIFFERENTIAL EVOLUTION

The DE algorithm uses a population set S . The initial set

$$S = \{x_1, x_2, \dots, x_N\}$$

consists of N random points in the search space. A contraction process is then used to drive these points to the vicinity of the global minimizer. The contraction process involves replacing bad point(s) in S with better point(s), per iteration. In particular, DE attempts to replace all points in S by new points at each iteration. DE, therefore, progresses in an epoch or era base. During each epoch k , N new function values are evaluated on N trial points. Trial points are found using mutation and crossover.

In each iteration, N competitions are held to determine the members of S for the next iteration. The i th ($i = 1, 2, \dots, N$) competition is held to replace x_i in S . Considering x_i as the target point a trial point y_i is found from two points (parents), the point x_i , i.e., the target point and the point \hat{x}_i determined by the mutation operation. In its mutation phase DE randomly selects three distinct points $x_{p(1)}$, $x_{p(2)}$ and $x_{p(3)}$, with replacement, from the current set S . None of these points should coincide with the current target point x_i . The weighted difference of any two points is then added to the third point which can be mathematically described as:

$$\hat{x}_i = x_{p(1)} + F(x_{p(2)} - x_{p(3)}), \quad (35)$$

where $F > 0$ is a scaling factor, and $x_{p(1)}$ is known as the base vector. If the point $\hat{x}_i \notin \Omega$ then the mutation operation is repeated. The trial point y_i is found from its parents x_i and \hat{x}_i using the following crossover rule:

$$y_i^j = \begin{cases} \hat{x}_i^j & \text{if } R^j \leq C_R \text{ or } j = I_i \\ x_i^j & \text{if } R^j > C_R \text{ and } j \neq I_i, \end{cases} \quad (36)$$

where I_i is a randomly chosen integer in the set I , i.e., $I_i \in I = \{1, 2, \dots, n\}$; the superscript j represents the j th component of respective vectors; $R^j \in (0, 1)$, drawn randomly for each j . The ultimate aim of the crossover rule (36) is to obtain the trial vector y_i with components coming from the components of target vector x_i and mutated vector \hat{x}_i . And this is ensured by introducing C_R and the set I . The targeting process continues until all members of S are considered. After all N trial points y_i have been generated, acceptance is applied. In the acceptance phase the function value at the trial point, $f(y_i)$, is compared to $f(x_i)$, the value at the target point. If $f(y_i) < f(x_i)$ then y_i replaces x_i in S , otherwise, S retains the original x_i . Reproduction (mutation and crossover) and acceptance continues until some stopping conditions are met. MDE differs from DE in that MDE uses a distribution to determine the value of F , and it uses some controlled localization of searches using the base vector for rapid convergence (Kaelo and Ali, 2006). DE uses a fixed value of F in (35). We now define the MDE algorithm.

ALGORITHM 2: The MDE Algorithm

Step 1. Determine the initial set

$$S = \{x_1, x_2, \dots, x_N\}$$

where the points $x_i, i = 1, 2, \dots, N$ are sampled randomly in Ω ; evaluate $f(x)$ at each $x_i, i = 1, 2, \dots, N$. Take $N \gg n$, n being the dimension of the function $f(x)$. Set iteration counter $k = 0$.

Step 2. Determine best, worst point in S . Determine the points x_{\max} and x_{\min} . If the stopping condition such as $|f_{\max} - f_{\min}| \leq \epsilon$ is satisfied, then stop. The points x_{\max} and x_{\min} and their function values f_{\max}, f_{\min} are such that

$$f_{\max} = \max_{x \in S} f(x) \quad \text{and} \quad f_{\min} = \min_{x \in S} f(x).$$

Step 3. Generate points to replace points in S for the next population (or iteration). For each $x_i \in S$ ($i = 1, 2, \dots, N$), determine y_i by the following two operations:

– **Mutation:**

$$\hat{x}_i = x_{tb} + F_i (x_{p(2)} - x_{p(3)}), \quad (37)$$

where x_{tb} is the tournament best (Kaelo and Ali, 2006) and $x_{p(2)}$ and $x_{p(3)}$ are the remaining two out of the three random vectors from S and $F_i \in [-1, -0.4] \cup [0.4, 1]$, chosen randomly. The tournament selection is applied for each i . If $\hat{x}_i \notin \Omega$ then we select another F_i .

– **Crossover:** Calculate the trial vector y_i corresponding to the target x_i from x_i and \hat{x}_i using the crossover rule (36).

Step 4. Acceptance rule to replace points in S . Select each trial vector y_i for the $k + 1$ iteration using the acceptance criterion: replace $x_i \in S$ with y_i if $f(y_i) < f(x_i)$ otherwise retain x_i . Set $k := k + 1$ and go to Step 2.

The DE algorithm (Kaelo and Ali, 2006; Storn and Price, 1997) does not require any properties of the function being optimized and it is easy to implement. Unlike the original DE algorithm, the MDE algorithm has only one parameter: the controlling factor in the crossover rule (36). We use the parameter values according to the suggestion made by (Ali and Törn, 2004).

5. Numerical Results

In this section we present the numerical results of the optimization problems. The parameter values of the optimization problems are interest, inflation and growth rates. The interest rate (z_1) and the inflation rate (z_2) are taken to be 12% per annum (pa) and 5% pa respectively. They are quoted by the *Reserve Bank of South Africa* (Reserve Bank, South Africa). The growth rate (z_3) for demand for services is found to be 6% pa (Ali, 2006). We use an annual growth rate (z_4) for tariffs as 3% pa. Notice that this rate is less than the inflation rate. This is compensated for by the growth in demand. All results are obtained for $d = 5$. The maximization problems are functions of the installation year. Therefore, the maximization is carried out with respect to PoP installation, i.e. which PoP is installed at which point in time over the payback period so as to maximize A_p^m and B_p^m , $m = 1, 2, 4, \dots, \infty$.

We used the ASA algorithm (Ali and Storey, 1997), presented in the previous section, to maximize the npv and *irr* defined by A_p^m and B_p^m , $m = 1, 2, 4$. These are all discrete variable problems. The constraints were satisfied by swapping. For example, if the first constraint in (33) was violated then we swap the position of installation between x_{10} and $\min\{x_1, x_6, x_9\}$.

We used the MDE algorithm (Kaelo and Ali, 2006) for the optimization problems involving continuous variables such as A_p^∞ and B_p^∞ . For the violation of constraints in the continuous problem a penalty value was used.

Numerical experiments with these problems have shown that except for A_p^∞ each of the other problems has a number of local minima. We have presented the best results of five independent runs. We have noticed that B_p^∞ is a more difficult problem than A_p^∞ . In particular, B_p^∞ needs higher number of function evaluations and cpu time to converge. Another difficulty associated with B_p^∞ is that of the occasional numerical instabilities in the solution of $F(irr)=0$. Numerical instabilities occur when the gradient of F is high and when an iterate of the solution process of $F(irr)=0$ produces a negative irr . We discard the rollout order for which the instabilities occur. For example, a rollout order that gives a high ($|F'(irr)| > 9 \times 10^3$) gradient of F is discarded and a different one is generated. Throughout the numerical calculation we have used $irr = 50\%$ as an initial guess to the solution of $F(irr)=0$.

All optimization problems involve calculation of subscription fees Y_k which depends on installation years. However, the contribution of Y_k in A_p^m and B_p^m can be separated. Once the final installation years are known Y_k can be calculated. At the final solution we exclude the contribution of Y_k from A_p^m and B_p^m in order to see the results of the individual PoP. In Table IV, we therefore present the optimal installation year and the corresponding net present value of profits ($P_{x_i}^v$) due to each PoP. The npvs in Table IV are in millions of Rands. We do not present the npv of profits of each PoP for B_p^∞ in Table IV. Results for B_p^∞ will be presented later. The last row in Table IV represents the total npv (npv_{pops}^v) due to PoPs only. It is clear from Table IV that the total npv is dominated by the npv for PoPs P_{10} , P_{12} , P_{16} and P_{25} throughout all problems. This is expected as one can see from Table III that these PoPs are where the majority of the users are concentrated. It can be also seen that the installation years of these PoPs are in the beginning of the first year of the payback period for A_p^∞ . The installation years for these PoPs have a different pattern for B_p^∞ . One can see that, unlike B_p^∞ , the optimal rollout order for A_p^∞ follows a bi-modal pattern, i.e. x_i are either at the beginning or at the end of the payback period. It is interesting to note that the total npv for A_p^∞ is smaller than those for A_p^2 and A_p^4 . All 5 runs for A_p^∞ obtained the same optimal value. For each optimal irr , we have also presented the corresponding npv at the bottom of the last three columns of Table 4. These npvs are less than the optimized npvs presented at the bottom of the columns 6–8 of the same table, as expected. Next we compare the cpu times needed to solve both problems. We present the cpu times in Table 5. These are the times required to obtain the results in Tables 4 and 8. It is clear from Table 5 that both

Table 4. Optimal rollout order and corresponding npv in millions of Rands

<i>i</i>	A_P^m				B_P^m				<i>i</i>	B_P^m					
	<i>m</i>				<i>m</i>					<i>m</i>					
	1	2	4	∞	1	2	4	∞		1	2	4	∞		
x_i	x_i	x_i	x_i	$P_{x_i}^v$	x_i	x_i	x_i	x_i	x_i	x_i	x_i	x_i	$P_{x_i}^v$		
1	2	6	7	1.26	-0.90	-0.33	1.49	-0.60	1	8	18	3.08	-0.97	-0.67	-0.55
2	3	3	12	4.99	-2.91	-7.23	-7.16	-1.13	3	9	10	3.45	-2.91	-1.87	-8.90
3	2	0	13	1.34	-0.91	-0.29	0.60	-0.62	3	5	19	0.93	-0.92	-0.24	-0.84
4	0	4	3	0.00	3.95	5.80	18.83	4.41	3	6	9	1.70	1.04	3.57	12.13
5	4	1	12	1.48	-1.01	-0.22	0.75	-0.64	2	6	10	0.59	-0.93	-0.36	1.06
6	0	0	1	0.00	3.99	10.04	21.11	4.46	2	7	18	3.91	2.09	2.44	1.36
7	4	5	11	4.99	-1.50	-2.77	-3.59	-1.13	4	9	18	1.35	-1.50	-1.39	-1.55
8	0	0	14	4.99	-3.25	-4.47	-1.99	-1.13	4	6	18	2.51	-1.40	-1.96	-1.35
9	1	3	6	0.00	0.63	2.70	7.17	1.36	1	9	17	2.35	0.63	-0.51	0.86
10	0	0	1	0.00	29.84	61.76	119.09	30.35	0	6	14	0.76	29.84	23.44	35.78
11	0	0	0	0.00	3.46	8.98	20.05	3.92	2	6	18	2.67	1.78	3.19	0.01
12	0	0	0	0.00	45.21	92.52	187.22	45.74	0	0	1	0.77	45.21	92.52	177.34
13	1	4	8	0.01	-0.56	0.47	2.61	-0.15	4	4	11	0.63	-0.90	0.47	1.89
14	4	8	10	4.99	-1.98	-2.63	-8.85	-1.13	2	9	12	3.88	-3.93	-1.86	-7.11
15	1	1	3	0.00	1.01	4.37	10.18	1.84	1	5	18	3.97	1.01	2.22	0.38
16	0	0	0	0.00	40.15	82.39	166.96	40.67	0	2	14	0.73	40.15	65.08	47.55
17	3	9	17	4.99	-2.71	-1.77	-2.92	-1.13	2	2	12	0.97	-3.63	-7.46	-6.33
18	1	2	2	0.00	3.07	8.00	20.04	4.45	4	4	16	1.68	-0.03	5.85	3.84
19	3	7	0	4.98	-1.53	-1.40	-4.78	-1.13	4	9	16	0.98	-1.30	-1.19	-1.24
20	0	3	3	0.00	0.88	2.68	8.50	1.34	4	7	11	4.63	-0.61	0.67	4.49
21	2	6	17	2.77	-1.22	-0.76	-0.60	-0.93	4	4	11	1.70	-1.10	-0.78	0.02
22	4	4	14	0.05	-0.91	0.41	0.96	-0.20	3	4	12	3.37	-0.73	0.41	1.54
23	1	4	18	4.99	-3.46	-3.81	-1.73	-1.13	4	6	9	2.38	-1.60	-2.75	-5.42
24	4	2	0	0.00	-0.71	2.29	7.66	0.83	4	3	15	0.82	-0.71	1.97	1.63
25	0	0	0	0.00	24.70	51.48	105.09	25.20	0	7	19	0.79	24.70	14.28	3.98
26	4	1	18	2.67	-1.10	-1.14	-0.76	-0.93	1	4	16	2.37	-1.41	-0.78	-0.45
27	1	8	3	0.07	-0.59	-0.49	3.28	-0.20	4	4	18	2.82	-0.91	0.41	-0.37
28	1	8	10	4.99	-3.43	-1.85	-4.86	-1.13	3	9	11	2.67	-2.11	-1.48	-4.39
29	0	4	19	0.14	-0.64	0.43	-0.75	-0.18	0	6	12	1.90	-0.64	0.04	1.57
30	2	9	3	0.00	0.89	-0.31	12.01	2.39	4	5	18	1.43	-0.42	2.74	0.59
31	4	1	14	0.00	-0.71	2.59	2.17	0.84	3	7	15	1.85	-0.32	0.38	1.65
32	4	7	19	3.99	-1.20	-1.09	-1.04	-1.09	4	8	16	2.92	-1.20	-1.08	-0.84
33	4	6	16	4.99	-1.30	-1.52	-1.20	-1.13	4	7	12	1.07	-1.29	-1.37	-1.55
34	4	9	12	4.99	-1.40	-1.29	-2.42	-1.13	3	5	11	2.99	-1.72	-2.26	-2.67
35	1	9	12	0.00	-0.16	-0.70	2.39	0.34	4	6	8	4.09	-0.80	0.45	3.79
36	4	7	13	4.99	-1.60	-2.26	-3.51	-1.13	2	5	6	0.82	-2.71	-3.21	-6.93
					122.04	300.58	672.01	150.17					109.66	188.90	250.97

algorithms required longer time to solve B_P^m than A_P^m . In particular, the cpu time of B_P^∞ is much higher than that of A_P^∞ . This shows that the problem B_P^∞ is a more difficult than A_P^∞ .

The npv for the i th PoP presented in Table 4 is its portion of contribution in the total npv_{pops}^v . The net present value of profits in each year generated by all PoPs can also be presented. For example, Table 6 presents

Table 5. cpu times required to obtain the results in Tables IV and VIII

m	1	2	3	4	∞
A_p^m	0.86	2.81	4.22	5.87	2.56
B_p^m	1.14	3.09	5.11	6.67	9.38

Table 6. Yearly cashflows in millions of Rands for A_p^1

i	CF_i^v	CF_i
0	-38.591	-38.591
1	5.449	6.103
2	23.093	28.968
3	28.908	40.614
4	4.854	7.638
5	98.335	173.300
	122.048	218.032

these values due to all PoPs for A_p^1 . In Table 6, i is year index, CF_i^v is the net present value of cashflow and CF_i is the cashflow at year i . It can be seen in Table 6 that the total npv of cashflow is 122.048 which is equal to the total npv due to all PoPs in Table 4 (see column 6). The *irr* of the cashflows CF_i is 64.98%. If we include the contribution of the subscription fees in CF_i then the *irr* becomes 64.83%. Both of these *irr* are less than the optimal *irr* (99.67%) in B_p^1 (see Table 9 later), as expected.

Before presenting the total npv and *irr* from the entire network, we describe how the Y_k values are obtained from a rollout order. For example, the optimal rollout order for A_p^1 is to install 10, 8, 4, 3 and 11 number of PoPs in year 0 to year 4 respectively (see column 2 in Table 4). The yearly usage for this rollout order is presented in Table 7. The data under year 0 represents the usage in year 0 from 10 PoPs. The data under year 1 is the total usage in year 1, i.e., 19967.601 gb is the result of a 6% pa increment on usage from the 10 PoPs installed in year 0 and 5285.541 gb is also an increased usage from 8 PoPs installed in year 1. The total yearly usage in the last row in this table shows that none of these usages is double the usage of the previous year. Therefore Y_k can easily be calculated using the total yearly usage from a table such as Table 7.

We now present the optimal rollout order and the cashflows generated for B_p^∞ . These are presented in Table 8 where c_{x_i} represents costs and r_{x_i} revenues generated at x_i . Therefore the cashflow $CF_i = r_{x_i} - c_{x_i}$, $i = 1, \dots, 37$. The cashflows CF_i , $i = 38, \dots, 42$, are due to the subscription

Table 7. Usage in gb per year due to optimal rollout order for A_p^1

year→	0	1	2	3	4
	18837.360	19967.601	21165.657	22435.596	23781.731
		5285.541	5602.673	5938.833	6295.126
			3527.609	3739.265	3963.620
				1671.671	1771.971
					9139.646
Total	18837.360	25253.142	30295.939	33785.365	44952.094

fees. The column under P_{r_i} represents the order at which the PoPs are installed. For example, the $P_{r_1} = 5$ th PoP was installed at first and the $P_{r_{36}} = 20$ th PoP was at last. The results for CF_i , r_{x_i} and c_{x_i} in Table 8 are in millions of Rands. The data under the column under x_i in Table 8 and in Table 4 for B_p^∞ are the same, except in Table 8 they are presented in the order at which the PoPs are installed. The i th cashflow presented in Table 8 is not the cashflow generated by the i th PoP. The i th ($i = 1, 2, \dots, 36$) cashflow is generated by the i th PoP installed at x_i and by all PoPs that are installed prior to the i th PoP. The resulting irr for the cashflows presented in Table 8 is 1353.31%. In the next table we present the optimal function values for all problems. It is clear from Table 9 that the npv f_p increases as m increases from 1 to 4. However, f_p decreases for $m = \infty$. In this particular problem, five independent runs produced the same minimum value. It is possible that each of these runs got trapped in a less than optimal solution. On the other hand, the irr increases with m for all values of m .

A comparison of the npvs in Tables 4 and 9 shows that the effect of the subscription fee is very little. However, numerical experiments have shown that B_p^m is very sensitive to the Y_k values. For example, if we double the yearly subscription fee X_0 then the optimal irr for B_p^∞ drops to 732.88% from 1353.31%. Besides, the number of function evaluations (in this case the solution of $F(irr) = 0$) increases by about 19%.

We now sum-up this section by noting that the mathematical model developed, for the purpose of assisting the investors, has been tested on a practical network problem. Results obtained in Table 9 shows that the optimization technique can potentially be used in assessing the commercial viability of a network project. For example, when the investors make an investment they expect an irr from the investment to make a profit. If the optimized irr is higher than their expected irr then the investors will receive higher returns on their investment, otherwise the investment will not be profitable to them. Therefore, the methodology presented here can guide the investors prior to any investment is made.

Table 8. Optimal rollout order vs cashflow in millions of Rands for B_p^∞

i	P_{r_i}	x_i	CF_i	i	CF_i	c_{x_i}	r_{x_i}
1	5	0.59	-1.65	38	0.00	1.657	0.000
2	13	0.63	-1.64	39	-0.05	1.742	0.094
3	16	0.73	0.68	40	-0.05	2.339	3.020
4	10	0.76	3.04	41	-0.05	3.113	6.153
5	12	0.77	7.03	42	-0.06	3.992	11.025
6	25	0.79	9.61			5.006	14.625
7	36	0.82	9.20			6.475	15.679
8	24	0.83	9.46			8.045	17.507
9	3	0.93	9.52			11.606	21.132
10	17	0.97	7.55			16.124	23.676
11	19	0.98	6.93			20.725	27.661
12	33	1.07	6.02			27.765	33.794
13	7	1.35	2.47			42.940	45.415
14	30	1.43	8.37			60.556	68.934
15	18	1.70	22.65			86.731	109.389
16	21	1.70	22.13			113.432	135.565
17	4	1.70	36.66			140.180	176.8413
18	31	1.85	42.95			173.067	216.026
19	29	1.90	46.10			208.253	254.357
20	9	2.35	60.50			265.027	325.536
21	26	2.37	60.07			322.932	383.007
22	23	2.38	44.44			381.136	425.578
23	8	2.51	33.80			446.577	480.379
24	11	2.67	72.93			521.478	594.412
25	28	2.67	53.39			596.475	649.865
26	27	2.82	62.41			681.146	743.559
27	32	2.92	57.96			772.202	830.164
28	34	2.99	43.09			868.480	911.573
29	1	3.08	49.62			971.181	1020.809
30	22	3.37	64.50			1096.572	1161.078
31	2	3.45	3.24			1228.293	1231.542
32	14	3.88	-73.59			1396.198	1322.602
33	6	3.91	31.08			1566.362	1597.444
34	15	3.97	91.58			1741.623	1833.213
35	35	4.09	127.57			1928.862	2056.441
36	20	4.63	201.98			2169.104	2371.088
37		5.00	88.44			277.325	365.768

Table 9. Optimal npv in millions of Rands and irr per cent from the entire network

$m \rightarrow$	A_p^m				B_p^m			
	f_p				irr			
	1	2	4	∞	1	2	4	∞
	121.86	300.39	671.92	150.17	99.67	458.71	1033.61	1353.31

6. Conclusion

We model two financial optimization problems and optimally determines the PoP rollout order of a network using the global optimization algorithms. This determines if the investors or the operators can achieve their profitability targets. When more than one architecture can be used in a certain PoP location, it is not clear upfront which architecture is the most profitable. The methodology presented here can easily handle this by reconfiguring the network. The optimization results can be used to configure a network from which test data can be generated. The test data can then be used to validate the network and therefore the design methodology. The solutions of the optimization problems can therefore provide an investment policy for investors in a data network. For instance, the optimization results such as *irr* can help the investors in adjusting the tariff to have a greater access in the competitive market. Therefore, our methodology can guide the investors. Furthermore, we have taken a combined telephone and data network, although our methodology can be applied separately to either.

The methodology can easily be extended to a decision engine in conjunction with the network design which will be capable of suggesting optimum network configurations to the network designer based on a range of hard and soft inputs such as the cost of the network, the ability to expand the network and the ability to meet the financial targets. Our ongoing research includes an integrated design methodology and software program that can be used to optimize a network.

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